Associative subalgebras of the Griess algebra and related topics

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1. Introduction

In [DMZ] it was shown that the moonshine module $V^\natural$ contains a sub vertex operator algebra isomorphic to the tensor product $L(\frac{1}{2},0)^{\otimes 48}$ where $L(\frac{1}{2},0)$ is the vertex operator algebra associated to highest weight unitary representation for the Virasoro algebra with central charge $\frac{1}{2}$. This containment turns out to be fundamental, because it allows us to deduce properties of $V^\natural$ from those of $L(\frac{1}{2},0)$, which are very much easier to discern (loc. cit.). This approach has been used to prove, for example, that $V^\natural$ is holomorphic [D], and to construct twisted sectors and intertwining algebras for $V^\natural$ [DLM], [H]. One can even base a simplified approach to the existence of $V^\natural$ on $L(\frac{1}{2},0)^{\otimes 48}$ (cf. Miyamoto's lecture at this conference).

The reason why $L(\frac{1}{2},0)$ is so attractive is that it is the first non-trivial *discrete series* representation of the Virasoro algebra and all modules and fusion rules are known for this family of vertex operator algebras. It is therefore a natural question to ask if $V^\natural$ contains other similar tensor products of discrete series representations, and more generally to ask which discrete series representations can be generated by idempotents in the Griess algebra B. We do not have complete answers to these questions, but we will take an approach which allows us, for example, to exhibit in a fairly painless way a sub vertex operator algebra of $V^\natural$ of the shape

$$\left(L(\frac{1}{2},0) \bigotimes L(\frac{7}{10},0) \bigotimes L(\frac{4}{5},0) \right)^{\otimes 12} \tag{1.1}$$

as well as another (of central charge less than 24) which is the tensor product of the first 24 members of the discrete series. It turns out that the key idea is to relate these questions to the theory of root systems and Niemeier lattices.

These issues are naturally related to the question of describing the maximal associative subalgebras of the Griess algebra. In this form, these question were first studied in [MN]. We show that each Niemeier lattice (and its attendant root system) determines (in many ways) certain maximal associative subalgebras of

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B. For example, (1.1) is associated to the Niemeier lattice of type A_2^{12}, while $L(\frac{1}{2}, 0)^{\otimes 48}$ corresponds to A_1^{24}.

2. Algebras and root systems

We consider a simple (i.e., irreducible) root system Φ of type A, D, E. Let Φ^+ be the set of positive roots and let $l \geq 1$ be the rank of Φ, $N = |\Phi^+|$, and h the Coxeter number of Φ. It is well-known that we have

$$2N = lh. \tag{2.1}$$

Any $\alpha \in \Phi^+$ determines a partition of Φ^+, namely

$$
\begin{aligned}
\Delta_0(\alpha) &= \{\alpha\} \\
\Delta_1(\alpha) &= \{\beta \in \Phi^+ | (\alpha, \beta) \neq 0, \beta \neq \alpha\} \\
\Delta_2(\alpha) &= \{\beta \in \Phi^+ | (\alpha, \beta) = 0, \beta \neq \alpha\}.
\end{aligned} \tag{2.2}
$$

Here, $(\cdot, \cdot)$ denotes the usual inner product associated with Φ, normalized so that $(\alpha, \alpha) = 2$ for $\alpha \in \Phi$. We often write $\alpha \sim \beta$ in case $\beta \in \Delta_1(\alpha)$; of course $\sim$ is a symmetric relation.

We will define a certain $\mathbb{Q}$-algebra $A = A(\Phi)$. Additively it is a free abelian group with a distinguished basis consisting of elements $t(\alpha)$, $u(\alpha)$ with $\alpha \in \Phi^+$. Thus A has rank $2N$. To define multiplication, note that if $\alpha \sim \beta$ with $\alpha, \beta \in \Phi^+$ then there is a unique $\gamma \in \Phi^+$ such that $\alpha \sim \gamma \sim \beta$. We use this observation to define

$$
t(\alpha)t(\beta) = \begin{cases} t(\alpha) + t(\beta) - t(\gamma), & \text{if } \alpha \sim \beta \sim \gamma \sim \alpha \\ 0, & \text{if } \beta \in \Delta_2(\alpha) \end{cases}
$$

$$
u(\alpha)u(\beta) = \begin{cases} u(\alpha) + u(\beta) - t(\gamma), & \text{if } \alpha \sim \beta \sim \gamma \sim \alpha \\ 0, & \text{if } \beta \in \Delta_2(\alpha) \end{cases}
$$

$$
u(\alpha)t(\beta) = t(\beta)u(\alpha) = \begin{cases} u(\alpha) + t(\beta) - u(\gamma), & \text{if } \alpha \sim \beta \sim \gamma \sim \alpha \\ 0, & \text{if } \beta \in \Delta_2(\alpha) \end{cases}
$$

$$
t(\alpha)^2 = 8t(\alpha), u(\alpha)^2 = 8u(\alpha).
$$

Obviously, $A = A(\Phi)$ is a commutative (but in general non-associative) algebra. The motivation for introducing this algebra will be explained later.

The following is well-known (cf. Chap. IV, 1.11, Proposition 32 of [B]).

LEMMA 2.1.   If $\alpha \in \Phi^+$ then $|\Delta_1(\alpha)| = 2h - 4$.

We use this to prove

PROPOSITION 2.2. $A(\Phi)$ has an identity element, namely

$$\delta = \frac{1}{4h} \sum_{\alpha \in \Phi^+} (t(\alpha) + u(\alpha)).$$

PROOF:—
If $\beta \in \Phi^+$ then $t(\beta)\delta$ is equal to

$$\frac{1}{4h} \left(t(\beta)^2 + t(\beta) \sum_{\alpha \in \Delta_1(\beta)} t(\alpha) + t(\beta) \sum_{\alpha \in \Delta_1(\beta)} u(\alpha) \right)$$

$$= \frac{1}{4h} \left(8t(\beta) + \sum_{\alpha \in \Delta_1(\beta)} (t(\beta) + t(\alpha) - t(\gamma(\alpha, \beta))) \right. \qquad (2.3)$$

$$\left. + \sum_{\alpha \in \Delta_1(\beta)} (t(\beta) + u(\alpha) - u(\gamma(\alpha, \beta))) \right)$$

where $\gamma(\alpha, \beta)$ is the element of Φ^+ determined by α and β whenever $\alpha \sim \beta$. But $\gamma(\alpha, \beta)$ ranges over $\Delta_1(\beta)$ as α does, so all terms in (2.3) cancel except for the $t(\beta)$'s. Lemma 2.1 tells us that (2.3) is thus equal to $\frac{1}{4h}(8t(\beta) + (2h - 4)t(\beta) + (2h - 4)t(\beta)) = t(\beta)$, that is $t(\beta)\delta = t(\beta)$. Similarly $u(\beta)\delta = u(\beta)$, and the proposition is proved.   □

We observe that the $\mathbb{Q}$-span of the $t(\alpha)$ for $\alpha \in \Phi^+$ is a subalgebra of A, which we denote by $T(\Phi)$. The same proof shows

LEMMA 2.3. $T(\Phi)$ has an identity, namely

$$\epsilon = \frac{1}{4h} \sum_{\alpha \in \Phi^+} t(\alpha).$$

Now introduce a symmetric bilinear form $\langle , \rangle$ on $A(\Phi)$ as follows:

$$\langle t(\alpha), t(\alpha) \rangle = \langle u(\alpha), u(\alpha) \rangle = 4$$

$$\langle t(\alpha), t(\beta) \rangle = \langle u(\alpha), u(\beta) \rangle = \begin{cases} 1/2, & \alpha \sim \beta \\ 0, & \alpha \in \Delta_2(\beta) \end{cases}$$

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$$\langle t(\alpha), u(\beta) \rangle = \begin{cases} 1/2, & \alpha \sim \beta \\ 0, & \alpha \in \Delta_0(\beta) \cup \Delta_2(\beta). \end{cases}$$

We remark that $\langle , \rangle$ is not necessarily non-degenerate (see below).

For an element $a \in A$ we define

$$c(a) = 8\langle a, a \rangle. \tag{2.4}$$

We may refer to $c(a)$ the *central charge* of a, though usually we reserve this term for the case that a is an idempotent of A such as δ or ϵ. In these cases we prove

LEMMA 2.4. *We have*
$$c(\delta) = l, \quad c(\epsilon) = \frac{lh}{h+2}.$$

PROOF:—

We have

$$c(\delta) = \frac{8}{(4h)^2} \sum_{\alpha \in \Phi^+} (\langle t(\alpha), t(\alpha) \rangle + \langle u(\alpha), u(\alpha) \rangle)$$

$$+ \frac{8}{(4h)^2} \sum_{\substack{\alpha \in \Phi^+ \\ \beta \in \Delta_1(\alpha)}} (\langle t(\alpha), t(\beta) \rangle + \langle u(\alpha), u(\beta) \rangle + \langle t(\alpha), u(\beta) \rangle + \langle u(\alpha), t(\beta) \rangle)$$

$$= \frac{1}{2h^2} (8N + N(2h-4)4)$$

$$= \frac{2N}{h}.$$

Now use (2.1) to see that $c(\delta) = l$. The calculation of $c(\epsilon)$ is similar. □

We now consider the possibility of decomposing the identity δ of A (or the identity ϵ of T) into a sum of idempotents $\delta = e_1 + e_2 + \cdots + e_k$ such that $\langle e_i, e_j \rangle = 0$ if $i \neq j$ and $e_i e_j = \delta_{i,j} e_i$. Such a decomposition corresponds to a particular kind of associative subalgebra of A isomorphic to a direct sum of k copies of $\mathbb{Q}$. One way to find such a decomposition is as follows: locate within the root system Φ a subsystem Φ', so that $A(\Phi')$ is a subalgebra of $A(\Phi)$. Then decompose δ as $\delta' + (\delta - \delta')$ where δ' is the identity of $A(\Phi')$. This has the desired properties (with $k = 2$). For large values of k one can iterate this procedure, considering chains of root systems $\Phi \supset \Phi' \supset \Phi'' \cdots$.

We illustrate this procedure in the case that Φ is of type A_l – the case of the most interest to us. Having fixed a system $\pi = \{\alpha_1, ..., \alpha_l\}$ of simple roots in Φ, we let Φ_i be the sub-system whose simple roots consist of $\alpha_1, ..., \alpha_i$, with A_i and T_i the corresponding algebras. Thus $A = A_l$ and $T = T_l$.

We begin by writing $\delta = \epsilon + (\delta - \epsilon)$ and verifying that indeed $\langle \epsilon, \delta - \epsilon \rangle = 0$, which we leave to the reader. Then we find that

$$c(\delta - \epsilon) = c(\delta) - (\epsilon) = \frac{2l}{h+2}.$$

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As Φ is of type A_l we have $h = l + 1$, so that

$$c(\delta - \epsilon) = \frac{2l}{l+3}. \tag{2.5}$$

Now consider the containment $T_{l-1} \subset T_l$ and let ϵ' be the identity of T_{l-1}, with l', N' having the obvious meaning.

LEMMA 2.5. *We have* $\langle \epsilon - \epsilon', \epsilon' \rangle = 0$.

PROOF:—
If $h' = h - 1$ is the Coxeter number of Φ_{l-1} then by Lemma 2.4 we have $\langle \epsilon, \epsilon' \rangle = \frac{(l-1)h'}{8(h'+2)}$. On the other hand we have

$$\langle \epsilon, \epsilon' \rangle = \frac{1}{(2h+4)(2h'+4)} \sum_{\alpha \in \Phi_l^+} \sum_{\beta \in \Phi_{l-1}^+} \langle t(\alpha), t(\beta) \rangle. \tag{2.6}$$

If $\alpha \in \Phi_{l-1}^+$ then $\sum_{\beta \in \Phi_{l-1}^+} \langle t(\alpha), t(\beta) \rangle = 4 + \frac{2h'-4}{2} = h' + 2$. If $\alpha \in \Phi_l^+ \backslash \Phi_{l-1}^+$ then $\sum_{\beta \in \Phi_{l-1}^+} \langle t(\alpha), t(\beta) \rangle = \frac{l-1}{2}$ since $|\Delta_1(\alpha) \cap \Phi_{l-1}^+| = l - 1$ for each α. Since $|\Phi_l^+ \backslash \Phi_{l-1}^+| = l$, we find that (2.6) yields

$$\begin{aligned}
\langle \epsilon, \epsilon' \rangle &= \frac{1}{(2h+4)(2h'+4)} \left(N'(h'+2) + \frac{l(l-1)}{2} \right) \\
&= \frac{1}{4(h'+2)(h'+3)} \left(\frac{h'l'(h'+2)}{2} + \frac{l(l-1)}{2} \right) \\
&= \frac{h'(h'+3)(l-1)}{8(h'+2)(h'+3)} \\
&= \frac{h'(l-1)}{8(h'+2)}
\end{aligned}.$$

So indeed $\langle \epsilon, \epsilon' \rangle = \langle e', e' \rangle$, as desired. $\qquad\square$

LEMMA 2.6. *We have* $c(e - e') = 1 - \frac{6}{(l+2)(l+3)}$.

PROOF:—
After Lemma 2.5 we have $c(\epsilon - \epsilon') = c(\epsilon) - c(\epsilon')$. Now use Lemma 2.4 to see that

$$\begin{aligned}
c(\epsilon - \epsilon') &= \frac{lh}{h+2} - \frac{l'h'}{h'+2} \\
&= \frac{l(l+1)h}{l+3} - \frac{l(l-1)}{l+2} \\
&= 1 - \frac{6}{(l+2)(l+3)}.
\end{aligned}$$

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If we iterate this procedure, we obtain the following result.

THEOREM 2.7. *Suppose that* $A = A(\Phi)$ *corresponding to the root system* Φ *of type* A_l. *Then the identity* δ *of* A *can be decomposed into a sum of* $l + 1$ *idempotents* $e_1, e_2, ..., e_{l+1}$ *satisfying the following*

(i) $\delta = e_1 + e_2 + \cdots e_{l+1}$

(ii) $e_i e_j = \delta_{ij} e_i$

(iii) $\langle e_i, e_j \rangle = 0$ *if* $i \neq j$.

(iv) $c(e_i) = 1 - \dfrac{6}{(i+2)(i+3)}$, $1 \leq i \leq l$.

(v) $c(e_{l+1}) = \dfrac{2l}{l+3}$.

REMARK. The significance of (iv), of course, is that the series of values $1 - \frac{6}{(i+2)(i+3)}$, $i = 1, 2, ...$ correspond to the central charges of the discrete series representations of the Virasoro algebra. These are the unitary highest weight representations of the Virasoro algebra with central charge c satisfying $0 < c < 1$. These values were first identified in [FQS], and the unitarity was proved in [GKO].

We owe to Jeff Harvey the observation that the central charge $\frac{2l}{l+3}$ also has some significance, namely it corresponds to the central charge of the parafermion algebras [ZF], [DL]. Both the discrete series and parafermionic representations of the Virasoro algebra arise from the GKO "coset construction," indeed our arguments leading to Theorem 2.7 amount to a coset construction at the level of root systems. We pursue these ideas in the following sections.

It goes without saying that (i)-(iii) and a modification of (iv) and (v) also hold if A_l is replaced by root systems of type D or E.

3. The vertex operator algebra $V_{\sqrt{2}R}^+$

Suppose that L is a positive-definite even lattice. The vertex operator algebra V_L associated with L is one of the most fundamental examples of a vertex operator algebra. We refer the reader to [FLM] for the construction of V_L. We wish to focus on the case in which R is the root lattice corresponding to a simple root system Φ as in Section 2, and where $L = \sqrt{2}R$ (later, we will relax our conditions to allow R to be semi-simple). Thus L is indeed a positive-definite even lattice which has, in addition, no vectors of squared length 2.

The lattice L has a canonical automorphism t of order 2, namely the reflection automorphism $t : x \mapsto -x$ for $x \in L$. Then t lifts to a canonical automorphism of V_L (loc. cit.), and we denote by V_L^+ the sub vertex operator algebra of t-fixed points of V_L. In the case that $L = \sqrt{2}R$, V_L^+ has no vectors of weight 1 owing to the absence of vectors in L of squared length 2. Thus

$$V_{\sqrt{2}R}^+ = \mathbb{C}\mathbf{1} \oplus B^+ \oplus \cdots \tag{3.1}$$

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where in (3.1) we have set $B^+ = (V_{\sqrt{2}R}^+)_2$, i.e., B^+ is the space of vectors in $V_{\sqrt{2}R}^+$ of weight 2.

There is a canonical structure of commutative, non-associative algebra on B^+, namely

$$ab = a_1 b, \ a, b \in B^+$$

where the vertex operator for a is given by

$$Y(a, z) = \sum_{n \in \mathbb{Z}} a_n z^{-n-1}.$$

Similarly, there is a non-degenerate, invariant inner product $\langle \cdot, \cdot \rangle$ on B^+ given by

$$\langle a, b \rangle \mathbf{1} = a_3 b.$$

Conveniently, the algebra structure of B^+ has been written down in Theorem 8.9.5 and Remark 8.9.7 of [FLM]. To describe it, let $H = \mathbb{Q} \otimes \sqrt{2}R$. Then

$$B^+ = S^2(H) \oplus \oplus_{\alpha \in \Phi^+} \mathbb{Q} x_\alpha$$

where

$$x_\alpha = e^{\sqrt{2}\alpha} + e^{-\sqrt{2}\alpha}$$

and $e^{\sqrt{2}\alpha}$ is the standard notation for the element in the group algebra $\mathbb{C}[L]$ corresponding to $\sqrt{2}\alpha$. (Note that in this case $\hat{L}$ is a direct product of L and $\langle \pm 1 \rangle$.) The relations are as follows:

$$h^2 \cdot k^2 = 4(h, k)hk, \quad h, k \in H$$
$$h^2 \cdot x_\alpha = 2(h, \alpha)^2 x_\alpha$$
$$x_\alpha \cdot x_\beta = \begin{cases} 0, & \beta \in \Delta_2(\alpha) \\ x_\gamma, & \alpha \sim \beta \sim \gamma \sim \alpha \\ 2\alpha^2, & \alpha = \beta. \end{cases}$$

Moreover

$$\langle h^2, k^2 \rangle = 2(h, k)^2$$
$$\langle h^2, x_\alpha \rangle = 0$$
$$\langle x_\alpha, x_\beta \rangle = \begin{cases} 0, & \alpha \neq \beta \\ 2, & \alpha = \beta. \end{cases}$$

THEOREM 3.1. *Let Φ be a simple root system of type A, D, E, let $A = A(\Phi)$ be as in Section 2, and let B^+ be as above. There there is an isometric surjection of algebras $A \to B^+$ given by*

$$\phi : t(\alpha) \mapsto \frac{1}{2}\alpha^2 - x_\alpha$$

$$u(\alpha) \mapsto \frac{1}{2}\alpha^2 + x_\alpha.$$

COROLLARY 3.2. *If Φ is of type A_l then the map $\Phi : A \to B^+$ is an isometric isomorphism of algebras*

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Note that $\dim B^+ = \frac{l(l+1)}{2} + N$. If Φ is of type A_l then $N = \frac{l(l+1)}{2}$, so that $\dim B^+ = 2N = \dim A$. So in this case, Theorem 3.1 implies the corollary.

REMARK. In the other cases, i.e., Φ of type D or E, we similarly see that the surjection $A \to B^+$ is *not* an isomorphism. Since it is an isometry (by the theorem), the kernel is precisely the radical of the form $\langle , \rangle$ on A.

PROOF OF THEOREM:—
This is quite straightforward. For example if $\alpha \sim \beta \sim \gamma \sim \alpha$ then

$$\phi(u(\alpha))\phi(t(\beta)) = (\frac{1}{2}\alpha^2 + x_\alpha)(\frac{1}{2}\beta^2 - x_\beta) = (\alpha, \beta)\alpha\beta - x_\gamma + \frac{1}{2}(\beta^2 x_\alpha - \alpha^2 x_\beta). \quad (3.2)$$

In this case we have $(\alpha, \beta) = -1$ and $\alpha\beta = \frac{1}{2}(\gamma^2 - \alpha^2 - \beta^2)$ (since $\alpha + \beta + \gamma = 0$). So (3.2) reads

$$\frac{1}{2}(\alpha^2 + \beta^2 - \gamma^2) - x_\gamma + x_\alpha - x_\beta = \phi(u(\alpha)) + \phi(t(\beta)) - \phi(u(\gamma))$$

as required. The other relations which show that ϕ is an algebra morphism are proved similarly. It is clear that ϕ is an epimorphism, so we only need check that it is also an isometry. This is also easy; for example

$$\langle \phi(u(\alpha)), \phi(t(\beta)) \rangle = \langle \frac{1}{2}\alpha^2 + x_\alpha, \frac{1}{2}\beta^2 - x_\beta \rangle = \frac{1}{2}(\alpha, \beta)^2 - 2\delta_{\alpha, \beta} = \langle t(\alpha), u(\beta) \rangle.$$

The other cases are similar. $\square$

Denote by $L(c, 0)$ the vertex operator algebra associated to the Virasoro algebra with central charge c. We have

THEOREM 3.3. *Let Φ be the simple root system of type A_l, with R the corresponding root lattice. Then the vertex operator algebra $V^+_{\sqrt{2}R}$ contains a sub vertex operator algebra isomorphic to a tensor product*

$$\bigotimes_{i=1}^{l+1} L(c_i, 0)$$

where $c_i = 1 - \frac{6}{(i+2)(i+3)}$ for $1 \le i \le l$ and $c_{l+1} = \frac{2l}{l+3}$.

PROOF:—
By a simple calculation, Miyamoto has shown (Theorem 4.1 of [M]) that if e is an idempotent in B^+ then $2e$ is such that the components of the vertex operator $Y(2e, z)$ generate a copy of the Virasoro algebra with central charge $c(e)$ in the sense of (2.4). Moreover, orthogonal idempotents e_i, e_j satisfying $e_i e_j = 0$ are such that the corresponding Virasoro algebras commute, that is they generate a tensor product of the two algebras (cf. Lemma 5.1 of (loc. cit.) and Lemma 2.4 of [DLM]). Now the theorem follows from Theorem 2.7. $\square$

REMARK. The theorem has an obvious extension to semi-simple root systems, all of whose simple components are of type A_l (varying l).

4. Niemeier lattices and the Moonshine module

Recall that a *Niemeier lattice*[*] is a self-dual, even, positive-definite lattice L of rank 24. The set of vectors L_2 of squared-length 2 forms a root system which is either empty (in which case L is the Leech lattice) or else itself has rank 24. In this latter case the root system L_2 is such that its simple components have a common Coxeter number. We call such a root system a Niemeier root system; then the map

$$L \to L_2$$

sets up a bijection between Niemeier lattices and Niemeier root systems (possibly empty). See [Nie] and [V] for background.

We now discuss the following result, which was hinted at in [Nor, P305] and which extends the results of [Cur].

THEOREM 4.1. *Let L be any Niemeier lattice. Then there is at least one (and in general several) isometric embedding $\sqrt{2}L \to \Lambda$, where Λ is the Leech lattice.*

We want to know how many sublattices of the Leech lattice Λ there are which are versions of each Niemeier lattice, including the Leech lattice itself, rescaled by a factor of 2. For each type of lattice the mass of sublattices of that type is shown in Table 1; the number of sublattices can be obtained from this by multiplying by the order of the Conway group Co_1, which is $2^{21}3^9 5^4 7^2 11.13.23$.

We now explain how the results in Table 1 may be obtained. First, it is easily seen (see Lemma 4.3 below) that if $N = \sqrt{2}L \subset \Lambda$ with L a Niemeier lattice then N corresponds to a Lagrangian (i.e., maximal totally isotropic) subspace of $\Lambda/2\Lambda$, under the orthogonal form whose value on a vector is 0 or 1 according as the squared length of the corresponding Leech lattice vector is or is not divisible by 4. This form is of type $+$, so that a Lagrangian subspace has dimension 12.

Now the number of extensions of any isotropic subspace to a Lagrangian one is known. To be precise, if the space has codimension n (i.e. dimension $12 - n$) the number will be

$$\prod_{i=0}^{n-1} (2^i + 1) \tag{4.1}$$

this being interpreted as 1 if $n = 0$. The problem now is to determine how to use this information to work out the number of Lagrangian subspaces of each type.

We do this by induction on the codimension. Let us call an isotropic subspace *connected* if it is generated by its minimal (norm 4) vectors, and these cannot be split into components that generate disjoint subspaces. It is well known that connected spaces correspond to Dynkin diagrams of type A_n ($n \geq 1$), D_n ($n \geq 4$) or E_n ($n = 6, 7, 8$). We classify all subspaces of each of these types and, for

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[*] Somewhat unconventionally, according to this terminology the Leech lattice falls under the rubric of Niemeier lattice.

each of them, determine how many extensions it has to connected spaces of the next higher dimension (or, equivalently, how many subspaces of the next lower dimension belong to each type.

This gives us the information we need. For by induction we already know how many Lagrangian spaces contain any connected subspace of higher dimension, and we know by (4.1) the total number of extensions. The difference between these will, therefore, be the number of extensions which contain a subspace of the relevant type as a connected component.

To take some examples, any Lagrangian subspace in which A_8 appears as a connected component must correspond to a Niemeier lattice of type A_8^3. If D_6 appears as a component then it will be of type D_6^4 or $A_9^2 D_6$; we can distinguish between these by counting the number of extensions of an A_9.

Table 1

Λ	153715/123771648
A_1^{24}	141985575/58032128
A_2^{12}	469525/19008
A_3^8	3077275/86016
A_4^6	39821/2560
$A_5^4 D_4$	24653/5120
D_4^6	58025/524288
A_6^4	16813/20160
$A_7^2 D_5^2$	6087/20480
A_8^3	1765/64512
$A_9^2 D_6$	4037/276480
D_6^4	791/294912
$A_{11} D_7 E_6$	67/46080
E_6^4	13/184320
A_{12}^2	61/366080
D_8^3	575/14680064
$A_{15} D_9$	41/3870720
$A_{17} E_7$	1/645120
$D_{10} E_7^2$	11/5160960
D_{12}^2	19/259522560
A_{24}	1/244823040
$D_{16} E_8$	1/660602880
E_8^3	1/1981808640
D_{24}	1/501397585920

The computations were done in **MAPLE.**

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Table 2 shows the complete list of connected isotropic subspaces. The first column shows the corresponding symbol, with Greek letters used to distinguish subspaces with the same Dynkin diagram. The second column shows the dimension. The third column gives the maximal intersection with the lattice of type A_{23} generated by vectors of type $(4, -4, 0^{22})$ (parametrized by the corresponding subset of the coordinate positions, on which the Conway group determines a Gloay code), plus the number of "spare" vectors. The symbols s_{12}^+, s_n and u_n are taken from [C1]: the first denotes the union of two special octads which intersect in a tetrad; the second a set of cardinality n in an ascending chain containing a special octad and the complement of another such, and *not* containing an s_{12}^+; and the third a set of the right size in an ascending chain linking a non-special hexad (i.e., *not* an s_6), a special dodecad, and the complement of a non-special hexad.

The fourth column shows the structure of the stabilizer of the subspace in Co_1. The subgroup to the left of the aligned column of dots is the pointwise stabilizer. (Square brackets denote an unspecified group of the relevant order.) The last column gives a list of lattices one dimension lower, with the number of extensions and containments of the relevant type. For example, the entry "$30{:}2(A_8^\alpha)$" in the row beginning "A_9^α" means that an A_9^α contains 2 A_8^α's, while an A_8^α extends to 30 A_9^α's.

Table 2

Symbol	Dim	Code	Stabilizer	Lattices
0	0		$Co_1 . 1$	
A_1	1	s_2	$Co_2 . 1$	$98280{:}1(0)$
A_2	2	s_3	$U_6(2). S_3$	$2300{:}3(A_1)$
A_3	3	s_4	$2^9.M_{21}. S_4$	$891{:}4(A_2)$
D_4	4	$s_4 + 1$	$2^{4+8}.A_5. 2^{1+4}.(S_3 \times 3)$	$21{:}12(A_3)$
A_4	4	s_5	$2^{1+8}.A_5. S_5$	$336{:}5(A_3)$
D_5	5	$s_5 + 1$	$2^7.2^4.3. 2^4.S_5$	$60{:}5(D_4), 5{:}16(A_4)$
A_5^α	5	s_6	$2^{1+8}.S_3. S_6$	$10{:}6(A_4)$
A_5^β	5	u_6	$2^6.3. S_6$	$160{:}6(A_4)$
E_6	6	$s_6 + 1$	$2^{1+8}.3. U_4(2).2$	$4{:}27(D_5), 1{:}36(A_5^\alpha)$
D_6^α	6	$s_6 + 1$	$2^6.2^4. 2^5.S_6$	$3{:}6(D_5), 3{:}32(A_5^\alpha)$
D_6^β	6	$u_6 + 1$	$2^6.3. 2^5.S_6$	$16{:}6(D_5), 1{:}32(A_5^\beta)$
A_7^α	6	s_8	$2^{1+8}. S_8$	$3{:}28(A_5^\alpha)$
A_6^α	6	u_7	$2^5. S_6$	$96{:}1(A_5^\alpha), 36{:}6(A_5^\beta)$
A_6^β	6	$u_6 + 1$	$3. S_7$	$64{:}7(A_5^\beta)$
E_7	7	$s_8 + 1$	$2^{1+8}. S_6(2)$	$3{:}28(E_6), 2{:}63(D_6^\alpha), 1{:}36(A_7^\alpha)$
D_8^α	7	$s_8 + 1$	$2^{1+8}.2^6.A_8$	$1{:}28(D_6^\alpha), 2{:}64(A_7^\alpha)$
D_7	7	$u_7 + 1$	$2^5.2^6.S_6$	$16{:}1(D_6^\alpha), 18{:}6(D_6^\beta), 1{:}64(A_6^\alpha)$
A_8^α	7	s_9	$2^4. A_8$	$64{:}1(A_7^\alpha), 2{:}28(A_6^\alpha)$
A_7^β	7	u_8	$2^4.2^4.S_4$	$30{:}8(A_6^\alpha)$
A_7^γ	7	$u_7 + 1$	$1. S_6 \times 2$	$32{:}2(A_6^\alpha), 63{:}6(A_6^\beta)$
E_8	8	$s_8 + 2$	$2^{1+8}. O_8^+(2)$	$1{:}120(E_7), 1{:}135(D_8^\alpha)$

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D_9^α	8	s_9+1	$2^4.2^7.A_8$	$16\!:\!1(D_9^\alpha), 1\!:\!28(D_7), 1\!:\!128(A_8^\alpha)$
D_8^β	8	u_8+1	$2^4.2^7.(2^4S_4)$	$15\!:\!8(D_7), 1\!:\!128(A_7^\beta)$
A_9^α	8	s_{10}	$2^3.2^4.L_3(2)$	$30\!:\!2(A_8^\alpha), 8\!:\!28(A_7^\beta)$
A_9^γ	8	s_9+1	$1.S_8$	$16\!:\!2(A_8^\alpha), 1\!:\!28(A_7^\gamma)$
A_8^β	8	u_9	$2^3.3^2.2S_4$	$16\!:\!9(A_7^\gamma)$
A_8^γ	8	u_8+1	$1.2^4.S_4$	$16\!:\!1(A_7^\beta), 30\!:\!8(A_7^\gamma)$
A_8^δ	8	u_7+2	$1.S_3 \wr S_3$	$10\!:\!9(A_7^\gamma)$
D_{10}^α	9	$s_{10}+1$	$2^3.2^8.(2^4.L_3(2))$	$15\!:\!2(D_9^\alpha), 4\!:\!28(D_8^\beta), 1\!:\!256(A_9^\alpha)$
D_9^β	9	u_9+1	$2^3.2^8.(3^2.2S_4)$	$8\!:\!9(D_8^\beta), 1\!:\!256(A_8^\beta)$
A_{11}^α	9	s_{12}^+	$2^2.2^6.3^{1+2}.2^2$	$14\!:\!18(A_9^\alpha), 8\!:\!64(A_8^\beta)$
A_{10}^α	9	$s_{10}+1$	$1.2^4.L_3(2)$	$8\!:\!1(A_9^\alpha), 30\!:\!2(A_9^\gamma), 4\!:\!28(A_8^\gamma)$
A_9^β	9	u_{10}	$2^2.S_6.2$	$6\!:\!10(A_8^\beta)$
A_9^δ	9	u_9+1	$1.3^2.2S_4$	$8\!:\!1(A_8^\beta), 8\!:\!9(A_8^\gamma)$
A_9^ϵ	9	u_8+2	$1.2^5.S_5$	$1\!:\!10(A_8^\gamma)$
A_9^ζ	9	u_8+2	$1.[2^63]$	$12\!:\!6(A_8^\gamma), 27\!:\!4(A_8^\delta)$
D_{12}^α	10	s_{12}^++1	$2^2.2^9.(2^6.3^{1+2}.2^2)$	$7\!:\!18(D_{10}^\alpha), 4\!:\!64(D_9^\beta), 1\!:\!512(A_{11}^\alpha)$
D_{10}^β	10	$u_{10}+1$	$2^2.2^9.S_6.2$	$3\!:\!10(D_9^\beta), 1\!:\!512(A_9^\beta)$
A_{15}^α	10	s_{16}	$2.2^4.A_8$	$6\!:\!140(A_{11}^\alpha), 4\!:\!448(A_9^\beta)$
A_{12}^α	10	s_{12}^++1	$1.2^6.3^{1+2}.2^2$	$4\!:\!1(A_{11}^\alpha), 7\!:\!18(A_{10}^\alpha), 4\!:\!64(A_9^\delta)$
A_{11}^β	10	u_{12}	$2.M_{12}$	$2\!:\!66(A_9^\beta)$
A_{11}^γ	10	$s_{10}+2$	$1.[2^93]$	$7\!:\!4(A_{10}^\alpha), 10\!:\!4(A_9^\epsilon), 3\!:\!24(A_9^\zeta)$
A_{10}^β	10	$u_{10}+1$	$1.S_6.2$	$4\!:\!1(A_9^\beta), 3\!:\!10(A_9^\delta)$
A_{10}^γ	10	u_9+2	$1.[2^43^2]$	$6\!:\!2(A_9^\delta), 12\!:\!9(A_9^\zeta)$
D_{16}	11	$s_{16}+1$	$2.2^{10}.(2^4.A_8)$	$3\!:\!140(D_{12}^\alpha), 2\!:\!448(D_{10}^\beta), 1\!:\!1024(A_{15}^\alpha)$
D_{12}^β	11	$u_{12}+1$	$2.2^{10}.M_{12}$	$1\!:\!66(D_{10}^\beta), 1\!:\!1024(A_{11}^\beta)$
A_{23}	11	s_{24}	$1.M_{24}$	$2\!:\!759(A_{15}^\alpha), 2\!:\!2576(A_{11}^\beta)$
A_{16}	11	$s_{16}+1$	$1.2^4.A_8$	$2\!:\!1(A_{15}^\alpha), 3\!:\!140(A_{12}^\alpha), 2\!:\!448(A_{10}^\beta)$
A_{13}	11	s_{12}^++2	$1.[2^93^2]$	$3\!:\!2(A_{12}^\alpha), 6\!:\!18(A_{11}^\gamma), 2\!:\!64(A_{10}^\gamma)$
A_{12}^β	11	$u_{12}+1$	$1.M_{12}$	$2\!:\!1(A_{11}^\beta), 1\!:\!66(A_{10}^\beta)$
A_{11}^δ	11	$u_{10}+2$	$1.S_6 \times 2$	$2\!:\!2(A_{10}^\beta), 1\!:\!10(A_{10}^\gamma)$
A_{11}^ϵ	11	u_9+3	$1.3^2.2S_4$	$4\!:\!12(A_{10}^\gamma)$
D_{24}	12	$s_{24}+1$	$1.2^{11}.M_{24}$	$1\!:\!759(D_{16}), 1\!:\!2576(D_{12}^\beta)$
A_{24}	12	$s_{24}+1$	$1.M_{24}$	$1\!:\!1(A_{23}), 1\!:\!759(A_{16})$
A_{17}	12	$s_{16}+2$	$1.2^4.A_8 \times 2$	$1\!:\!2(A_{16}), 1\!:\!140(A_{13}), 1\!:\!448(A_{11}^\delta)$
A_{15}^β	12	s_{12}^++4	$1.[2^{12}3^3]$	$1\!:\!24(A_{13}), 1\!:\!256(A_{11}^\epsilon)$
A_{12}^γ	12	u_9+4	$1.L_3(3)$	$1\!:\!13(A_{11}^\epsilon)$

Let us fix a Niemeier lattice L *not* equal to the Leech lattice, and let $R = L_2$. So there are isometric embeddings

$$\sqrt{2}R \to \sqrt{2}L \to \Lambda$$

and correspondingly there are vertex operator algebra embeddings

$$V^+_{\sqrt{2}R} \to V^+_{\sqrt{2}L} \to V^+_{\Lambda} \to V^\natural \tag{4.2}$$

where $V^\natural$ is the moonshine module [FLM].

Suppose that L_2 is one of the following type: A_1^{24}, A_2^{12}, or A_{24}. Then we may use (4.2) together with Theorem 3.3 to see that $V^\natural$ contains sub vertex operator algebras of the following kind:

$$L(\frac{1}{2}, 0)^{\otimes 48} \tag{4.3}$$

$$\bigotimes_{i=1}^{24} L(c_i, 0) \bigotimes L(\frac{16}{9}, 0) \tag{4.4}$$

as well as the type in equation (1.1).

Type (4.3) was already constructed in [DMZ]. Of course, we get many analogous tensor products by using other Niemeier lattices. Type (1.1) is of interest because, along with (4.3), it seems to be the only tensor product of Virasoro algebras that one can obtain in this way which is both of central charge 24 and has only discrete series as factors. The factor $L(\frac{116}{117}, 0)$ in (4.4) (corresponding to $i = 24$) corresponds to the discrete series with largest value of c which we know occurs via an idempotent of the Griess algebra. It would be interesting to know if it is indeed the maximal such value of c.

Of course, the embeddings (4.2) imply embeddings of the corresponding homogeneous spaces. In particular, if we let $B^+(L_2)$ be the weight 2 subspace of $V^+_{\sqrt{2}R}$ then we get $B^+(L_2) \subset B$. Then application of Theorem 3.3 (and the remark following it) yields embeddings of associative subalgebras into the Griess algebra B. For example if L_2 is of type A_1^{24}, corresponding to type (4.3), we get a maximal associative subalgebra of B of dimension 48. This was first constructed in [MN]. Similarly type (1.1) gives a maximal associative subalgebra of B of dimension 36. In the general case we have

LEMMA 4.2. *Let L be a Niemeier lattice with root system L_2 which has k simple components. Then L determines (in several ways) an associative subalgebra of B of dimension $24 + k$.*

PROOF:—

The procedure of Theorem 2.7 yields an $(l + 1)$-dimensional associative algebra for any simple root system of rank l, and it maps isomorphically into the

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corresponding B^+ of theorem 3.1 because ϕ is an isometry. Thus our Niemeier lattice L affords an associative algebra of dimension $\sum_{i=1}^{k}(1+l_i)$ where $\{l_i\}$ are the ranks of the simple components of L_2. Since $\sum_i l_i = 24$, the lemma follows. $\square$

There is another way to look at these matters which involves the Monster more directly and which was briefly touched on above and in [Nor]. Namely, let $\bar{\Lambda} = \Lambda/2\Lambda$ be the Leech lattice mod 2 equipped with the non-degenerate form which was described earlier. Then $\bar{\Lambda}$ has type +, that is the Lagrangian subspaces have rank 12. If $\bar{U} \subset \bar{\Lambda}$ is such a space then the full inverse image $U \subset \Lambda$ has index 2^{12}, so that $\frac{1}{\sqrt{2}}U = L$, say, is a unimodular lattice. Moreover since $\bar{U}$ is totally isotropic then L is integral and hence is a Niemeier lattice. Thus again we see how to embed $U = \sqrt{2}L \to \Lambda$. Indeed, for a Niemeier lattice L together with an isometric embedding $\sqrt{2}L \to \Lambda$ we have already observed that $|\Lambda : \sqrt{2}L| = 2^{12}$ and $\sqrt{2}L$ maps to a totally isotropic subspace of $\bar{\Lambda}$, hence is Lagrangian. We have shown

LEMMA 4.3. *There is a natural bijection between embedded and re-scaled Niemeier lattices $\sqrt{2}L \to \Lambda$ and Lagrangian subspaces of $\Lambda/2\Lambda$.*

Next we lift $\Lambda/2\Lambda$ to an extra-special group $Q \simeq 2^{1+24}_+$. As is well-known [G], the subgroup C of the Monster which leaves invariant the "untwisted" part V_Λ^+ of $V^\natural$ (cf. (4.1)) contains Q as a normal subgroup with quotient $C/Q \simeq Co_1$.

The Lagrangian subspaces of $\bar{\Lambda}$ are precisely those which lift to (maximal) elementary abelian subgroups of Q isomorphic to $\mathbb{Z}_2^{13}$. Now in the Monster there are two classes of involutions, of types $2A$ $(2+)$ and $2B$ $(2-)$ respectively, and one may ask how they distribute themselves among the $\mathbb{Z}_2^{13}$ subgroups. There is a pretty answer, which runs as follows: one knows (cf. [C1], for example) that the $2A$ involutions of Q map onto elements of $\Lambda/2\Lambda$ which themselves lift to elements $\lambda \in \Lambda$ satisfying $(\lambda, \lambda) = 4$, whereas the $2B$ involutions correspond to λ such that $(\lambda, \lambda) = 8$. Thus for a fixed maximal elementary abelian subgroup $E \leq Q$ corresponding to the re-scaled Niemeier $\sqrt{2}L \subset \Lambda$, the elements in E of type $2A$ correspond precisely to the elements of squared-length 2, i.e., to the elements of L_2. Of course, these form a Niemeier root system. So we have proved

PROPOSITION 4.4. *Let $E \subset Q$ be a maximal elementary abelian subgroup which corresponds (via the bijection of Lemma 4.3) to the re-scaled Niemeier lattice $\sqrt{2}L$. Then there is a natural bijection*

$$\{2A \text{ involutions in } E\} \to \text{root system of type } L_2.$$

Let E be as in Proposition 4.4, corresponding to $\sqrt{2}L$. We come full circle with the next result, which identifies the sub vertex operator algebra of $V^\natural$ fixed pointwise by E.

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PROPOSITION 4.5. *There is a natural isomorphism of vertex operator algebras*

$$(V^\natural)^E \simeq V^+_{\sqrt{2}L}.$$

PROOF:—
   Consider V_Λ, which is linearly isomorphic to $S(H_-) \otimes \mathbb{C}[\Lambda]$ (cf. [FLM]) where $H = \mathbb{C} \otimes \mathbb{Z}$, $H_- = H_{-1} \oplus H_{-2} \oplus \cdots$ with each $H_i \simeq H$, and $\mathbb{C}[\Lambda]$ is the group algebra of Λ. Now the center $Z(Q)$ of Q acts on $V^\natural$ with fixed-point sub vertex operator algebra naturally isomorphic to V^+_Λ, so we can study the action of $E/Z(Q)$ on V^+_Λ to prove the Proposition.
   Now elements γ of $\mathbb{R} \otimes_{\mathbb{Z}} \Lambda$ act on V_Λ by fixing $S(H_-)$ identically and acting on basis vectors e^α, $\alpha \in \Lambda$, via

$$\gamma \cdot e^\alpha = e^{2\pi i (\gamma, \alpha)} e^\alpha.$$

This induces an action of the Leech torus $\mathbb{R}^{24}/\Lambda$ on V_Λ in which E corresponds to $(\sqrt{2}L)^*/\Lambda = \frac{1}{\sqrt{2}}L/\Lambda$. Thus if $\alpha \in \Lambda$, $\gamma \cdot e^\alpha = e^\alpha$ for all $\gamma \in \frac{1}{\sqrt{2}}L$ if, and only if, $\alpha \in \sqrt{2}L$.
   This shows that $V_\Lambda^{\frac{1}{\sqrt{2}}L/\Lambda} \simeq V_{\sqrt{2}L}$, and therefore also

$$(V^\natural)^E \simeq (V^+_\Lambda)^{\frac{1}{\sqrt{2}}L/\Lambda} \simeq (V_\Lambda^{\frac{1}{\sqrt{2}}L/\Lambda})^+ \simeq V^+_{\sqrt{2}L}.$$

$\square$

   Finally, let us continue to let E and L be as above. Restricting Proposition 4.5 to the Griess algebra yields an isomorphism of algebras

$$B^E \simeq (V^+_{\sqrt{2}L})_2$$

and in particular $B^E \supset B^+(L_2) = (V^+_{\sqrt{2}L})_2$ where R is the root lattice L_2. This refines the containment of $B^+(L_2)$ in B found earlier.
   We have seen in Theorem 3.1 (and its extension to semi-simple root systems) that there is a surjection of $A(L_2)$ onto $B^+(L_2)$ and in particular $B^+(L_2)$ is generated by the images of $t(\alpha), u(\alpha)$ for $\alpha \in L_2^+$.
   Now as $t(\alpha)^2 = 8t(\alpha)$ and $\langle t(\alpha), t(\alpha)\rangle = 4$ then each $t(\alpha)/8$ corresponds to an idempotent with central charge $1/2$ (cf. (2.4)). Of course α itself corresponds to an involution of type $2A$. It turns out that the bijection $\alpha \mapsto t(\alpha)/8$ is precisely the correspondence between *transpositions* (i.e., $2A$ involutions of the Monster) and *transposition axes* in the Griess algebra. See [C2] and [M] for more information on this point. Using this perspective, one can read off the relations satisfied by the images of the $t(\alpha)/8$, from Table 1 of [Nor]. This was the original motivation for introducing the algebra $A(\Phi)$.

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